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Temporal and structural heterogeneities emerging in adaptive temporal networks

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We introduce a model of adaptive temporal networks whose evolution is regulated by an interplay between node activity and dynamic exchange of information through links. We study the model by using a master equation approach. Starting from a homogeneous initial configuration, we show that temporal and structural heterogeneities, characteristic of real-world networks, spontaneously emerge. This theoretically tractable model thus contributes to the understanding of the dynamics of human activity and interaction networks.

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Human social behavior depends both on intrinsic properties of the individuals and on the interactions between them. In daily life, interactions between people create contact patterns that can be mathematically represented by networks, i.e., a set of nodes, corresponding to the people, connected by links, representing the contacts between the respective individuals [1]. In network terminology, the number of contacts of a given person is called the degree k of a node and thus the degree distribution p_k is the probability distribution that a randomly chosen node has degree k. A central observation is that most real-life networks have a high level of heterogeneity in the number of contacts per node and the empirical degree distributions are typically approximated by power laws $p_k \sim k^{-\gamma}$ [2–4].

A second observation is that human contact networks are not static. For instance, in studies of email contact networks, users that are hubs of the network in one time window may be unremarkable or even isolated in the next time window [5]. This may reflect the fact that nodes alternate between an active, contact-seeking state, and an inactive state. The temporal heterogeneity can then be quantified in terms of the interevent intervals (IETs) between node activations, which reveals burstiness of human behavior [6,7].

While several models have been proposed to explain the heterogeneity in the degree distribution [8–11], fewer studies focused on modeling the burstiness of the temporal activity [6,7,12]. Barabási proposed a priority-based model in which nodes first execute the high-priority tasks, i.e., these tasks are executed within a short time, while low-priority tasks have to wait longer times before leaving the queue. Other models use inhomogeneous Poisson processes on each node modulated by (daily and weekly) cycles of human activity [13,14]. Combined, these processes generate the patterns of burstiness that are comparable to real-world data.

While previous models thus describe network heterogeneity and burstiness as different phenomena, they are connected in the real world: Network structure is a result of the activity of the network nodes, while changes in activity are likely to be triggered by neighbors in the network. The system is thus modeled as an adaptive network [15,16], where dynamics of nodes is affected by network structure, while the evolution of the structure is dependent on the state of the nodes.

A number of models have been proposed [17,18], in which the network links are temporal and change according to the state of nodes [19,20]. In particular, an idea that temporal links are generated according to the activity parameter of nodes that is obtained by empirical data was independently presented in Refs. [21,22]. This modeling framework is extended to incorporate the memory effect of the past contacts [23,24], which can be also regarded as a type of adaptive network.

In this Rapid Communication, using the framework of Refs. [21,22], we consider an interplay between individual activity and network structure: the human activity is dynamically determined by the state of the node, and simultaneously the state is updated by the contacts between the nodes, as illustrated in Fig. 1(a). Our toy model is motivated by interactions in online environments, particularly web forums [25] and dating sites [26], where there is an interplay between exchange of information (i.e., messages) and activity (i.e., posting) of the members, such that past interactions trigger new events. We analyze the master equation of the model using generating functions. We show that, starting from homogeneous initial conditions, structural heterogeneity and temporal burstiness can emerge spontaneously.

We consider a population of N nodes, where each node i has an intrinsic variable x_i, which we interpret as an abstract resource representing the node's willingness to engage with others in the network. Initially every node is assigned the same resource quantity, i.e., $x_i(0) = 1$. The system then evolves due to dynamical updates which comprise three steps: (i) activation of nodes, (ii) formation of pairs, and (iii) exchange of resources [Fig. 1(b)].

In the activation step (i) N_A nodes are set to the active state, while all others are set to the inactive state. The active nodes are chosen initially using a linear preferential rule, such that the probability that node i becomes active scales with x_i. In the pair formation step (ii) every active node picks a partner. With probability κ this partner is chosen randomly from all nodes in the system. With the complementary probability $1-\kappa$ the partner is chosen randomly from the set of active nodes. This also reduces to previously studied rules in the limit $\kappa = 0$ [21],

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RAPID COMMUNICATIONS

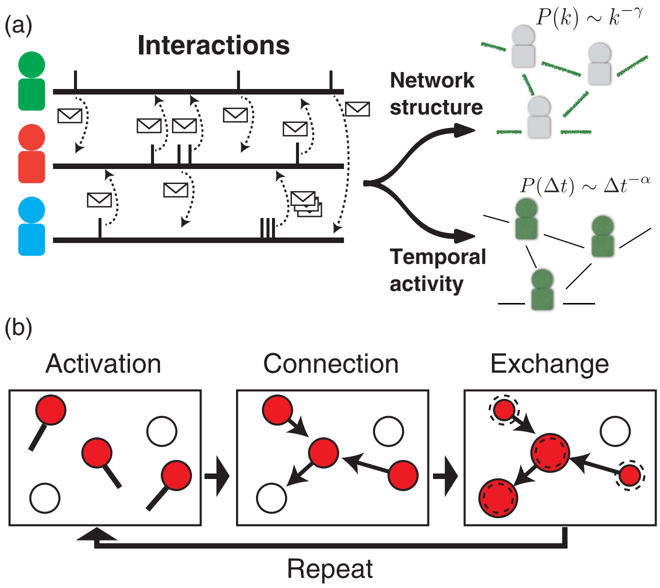

FIG. 1. (a) Question: How do social interactions induce structural and temporal heterogeneities among people? (b) Illustration of the interaction-regulated stochastic contact model. Within one time step, (i) nodes become activated, (ii) make random connections, (iii) exchange resources, and finally (iv) break down the links.

where partners are only picked among the active nodes, and in the limit $\kappa = 1$ [22], where partners are picked among the whole network. Finally, in the exchange step nodes transfer an amount of resource to their partners such that

$$x_i(t+1) - x_i(t) = D\left[\sum_j a_{ij}(t) - \sum_j a_{ji}(t)\right], \quad (1)$$

where D is the amount of resource transferred in the interaction, and $a_{ij}(t) = 1$ if i has picked j as its partner, and 0 otherwise.

Throughout the above procedure, this model mimics social interactions in online, web forums and social networking services. Users become active according to their willingness to engage with others denoted by x_i, and either comment on someone else's threads (random connections over the network) or respond to posts of other active (connections to active nodes) members. This will increase the willingness of receivers who get the comments, whereas it will satisfy the demand of the senders (resource exchange).

First, we demonstrate the self-organization of the system. We consider the model from a network perspective. Node i picking a partner j constitutes the creation of a directed link from i to j at every step. Thus the matrix $\mathbf{a(t)}$ is interpreted as a directed adjacency matrix. As a result of the temporal contacts between nodes, the initial identical distribution of resources on nodes becomes heterogeneous over time. This leads to the emergence of a dynamic network with structural heterogeneity and burstiness of node activations, as shown in Fig. 2(a) (see movie for a dynamic behavior of the model [27]).

By tuning the parameter κ, this model is able to reproduce different structural and temporal patterns (Fig. 2). In the figure, we show the degree distribution of the aggregated network

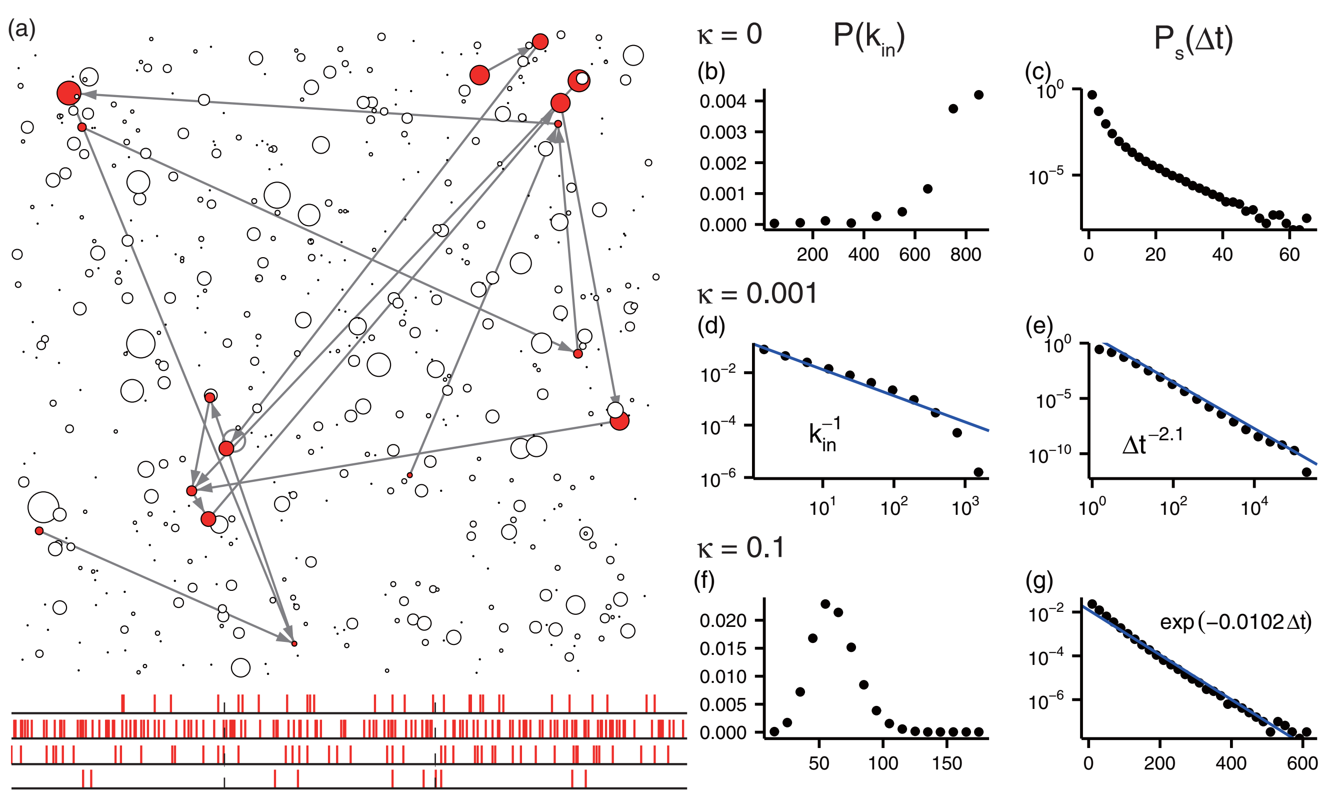

FIG. 2. (a) A snapshot of dynamic behavior of the model (see movie in Supplemental Material for details). (b)–(g) The degree distribution $P(k)$ and single-node IETs distribution $P_s(\Delta t)$ for [(b),(c)] $\kappa = 0$ [$P_s(\Delta t)$ is plotted in a semilogarithmic graph], [(d),(e)] $\kappa = 0.001$ (both graphs are log-log), and [(f),(g)] $\kappa = 0.1$ [$P_s(\Delta t)$ is semilogarithmic], with $N_A = 1024$, $N = 2^{15}$, and $D = 0.01$.

which is formed by all links collected during $T_s = 10^4$ time steps, and the IET distribution of a single node during 5×10^7 steps. If $\kappa = 0$, a small group of active nodes emerges. Nodes outside this group are left without resources. In this situation, the IET distribution of the active nodes exhibits a Poisson-like dynamics, i.e., exponential interevent times [Fig. 2(c)]. By contrast, sufficiently large κ leads to a homogeneous distribution of resources, which generates a Gaussian-like in-degree distribution [Fig. 2(f)] and an exponential (single-node) IET distribution [Fig. 2(g)], the latter a result of the quasihomogeneous Poisson process. In the intermediate case, where active nodes mainly link to other active individuals but also occasionally inactive ones, we observe the emergence of highly heterogeneous structure [Fig. 2(d)] and temporal patterns [Fig. 2(e)]. Owing to the period for aggregation $T_s(= 10^4)$, the network has an upper limit of the degree and its distribution has an exponential cutoff. In this situation, the distributions of the weights and the IETs at a link level are also very heterogeneous, and can be fitted by a power law (data not shown).

To make theoretical progress we construct a master equation for the resource dynamics. We define $u_n(t)$ as the density of nodes i at resource level $x_i = nD$. Setting the total resource in the system to $\sum nDu_n = 1$ and assuming large N and N_A, the probability that a node at resource level n becomes active is $a_n = (N_A/N)(nD/\sum_n nDu_n) = nDN_A/N$. Therefore, the proportion of nodes that are both active and at resource level n is $a_n u_n = N_A nDu_n/N$.

Since the total number of active nodes is N_A, N_A links are formed in every time step. Of these, $M1 := N_A\kappa$ links point to random targets chosen among the whole population, whereas $M2 := N_A(1-\kappa)$ point to targets chosen only among the active nodes. From the perspective of a single node the placement of a link can be seen as a statistical trial that is successful if that specific node is chosen as the target of the link. Every node, irrespective of state, receives a link with probability $\rho_1 = 1/N$ in each of the M_1 trials where the targets are random nodes. In addition, active nodes receive a link with probability $\rho_2 = 1/N_A$ in each of the M_2 trials where the targets are random active nodes.

Each node then gains D units of resource for every incoming link, while active nodes lose D units of resource via their outgoing link. Using the binomial distribution $B(m,\rho,M)$ of m success in M trials, then leads to the master equation

$$\begin{aligned}\frac{du_n}{dt} &= A(-1)a_{n+1}u_{n+1}\\ &\quad -C_1 a_n u_n - C_2\bar{a}_n u_n\\ &\quad +\sum_{m=1}^{N_A} a_{n-m}u_{n-m}A(m)\\ &\quad +\sum_{m=1}^{N_A\kappa} \bar{a}_{n-m}u_{n-m}B(m,\rho_1,M_1),\end{aligned} \tag{2}$$

where we now treat time continuously and $\bar{a}_n(= 1 - a_n)$ is the inactive fraction of $u_n(t)$, $A(m) = \sum_{m_1+m_2=m+1} B(m_1,\rho_1,M_1)B(m_2,\rho_2,M_2)$, and $C_1 = 1 - A(0)$, $C_2 = \sum_{m=1}^{N_A\kappa} B(m_1,\rho_1,N_A\kappa)$.

For the analysis of the master equation it is useful to write a generating function $Q(t,x) = \sum_n u_n(t)x^n$ [27,28]. This function encodes the u_n in a continuous function by interpreting them as Taylor coefficients in an abstract variable x, which does not have a physical meaning. Multiplying Eq. (2) by x^n and summing over $n \geqslant 0$ we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \frac{N_A D}{N}\frac{\partial Q}{\partial x}\left[A(-1) + (-C_1 + C_2)x + \sum_{m=1}^{N_A-1} A(m)x^{m+1}\right.\\ &\quad \left. -\sum_{m=1}^{N_A\kappa} B(m,\rho_1,M_1)x^{m+1}\right]\\ &\quad +Q\left[-C_2 + \sum_{m=1}^{N_A\kappa} B(m,\rho_1,M_1)x^m\right].\end{aligned} \tag{3}$$

In the limit $N,N_A \to \infty$, we approximate the binomial distributions reduce to Poisson distributions with finite rates, $\lambda_1(= \rho_1 M_1 = \frac{N_A}{N}\kappa)$ and $\lambda_2(= \rho_2 M_2 = 1 - \kappa)$, respectively. We obtain

$$\frac{\partial Q}{\partial t} = \frac{N_A D}{N} Y(x)\frac{\partial Q}{\partial x} + Z(x)Q, \tag{4}$$

where $Y(x) = \exp[(\lambda_1 + \lambda_2)(x-1)] - x\exp[\lambda_1(x-1)]$ and $Z(x) = -1 + \exp[\lambda_1(x-1)]$. Thus the generating approach has converted the large system of ordinary differential equations to a single partial differential equation.

When considered in the stationary state, Eq. (4) relates Q to its own first derivative Q'. For any probability distribution the corresponding generating function must obey $Q(1) = 1$. We can therefore find $Q'(1)$ and by differentiating $Q''(1)$. These quantities are of interest because $Q'(1) = \mu$ is the mean of u_n and $Q''(1)$ is closely related to the variance $\sigma^2 = Q''(1) + Q'(1) - Q'(1)^2$ of u_n. From this we can obtain the mean μ_x and variance σ_x^2 of the resource distribution

$$\mu_x = 1, \tag{5}$$

$$\sigma_x^2 = \frac{D}{2\kappa}\left[1 - 2\kappa + \left(1 - \frac{N_A}{N}\right)\kappa^2\right] + D. \tag{6}$$

In the case where targets are almost always chosen among the active nodes ($\kappa \sim 0$), we find the resource amounts among nodes will be extremely heterogeneous [Fig. 3(a)]. This result is also found in agent-based simulations as shown in the inset of Fig. 3(a), in which the resource distribution is well fitted by a power law x^{-1} in some ranges. We further confirmed that the same scaling behavior also appears in a continuous-time version of the model (not shown). These results are interesting since the power-law exponent $\gamma = 1$ differs from $2 < \gamma < 3$ that is reported to most scale-free networks in empirical studies [1,29]. At present we do not have an explanation for this exponent and cannot strictly exclude that the $1/k$ behavior is an extremely long transient.

The master equation loses information on the temporal activity of the nodes. Instead of single-node IETs $P_s(\Delta t)$, we thus consider the population IETs $P_e(\Delta t)$. If the amount of resources is fixed, the activation of u_n^* can be described by a Poisson process with a fixed rate a_n. In general, the population IETs of homogeneous Poisson nodes $\{p_i\}$ $(i = 1,2,\ldots,N)$ is

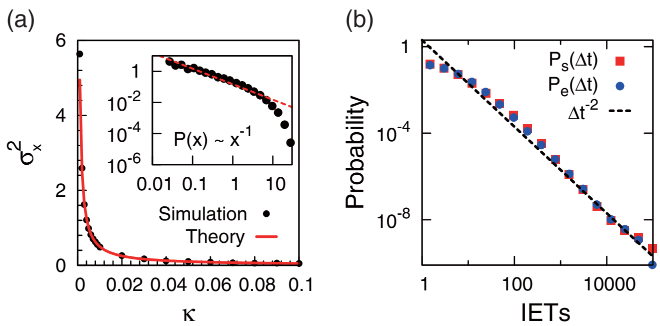

FIG. 3. (a) The variance σ_x^2 of the stationary resource distribution given by Eq. (6) (red line) and for the simulation of the agent-based model (black circles). We use $D = 0.01$, $N_A = 1000$, and $N = 2^{15}$. The inset shows the resource distribution for $\kappa = 0.001$. (b) The distributions of population and single-node IETs, respectively, $P_e(\Delta t)$ and $P_s(\Delta t)$. The master equation analysis predicts $P_e(\Delta t) \propto \Delta t^{-2}$.

given by [14]

$$P_e(\Delta t) = \sum_i p_i e^{-p\Delta t} = \int f(p) p e^{-p\Delta t} dp,$$

where $f(p)$ is the rate distribution in the limit of $N \to \infty$. In this equation, only the first time interval of the node's activations is collected for each node [14]. The second, third, and succeeding intervals should be taken into account for population IETs during a given observation period. The higher the rate of a node, the more likely the IET data is collected from this node. The probability of the IET should be multiplied by the rate p, and then the equation is rewritten as

$$P_e(\Delta t) = \int f(p) p^2 e^{-p\Delta t} dp \propto \Delta t^{-2}. \quad (7)$$

In the last part, we considered the case of $f(p) \propto p^{-1}$, because u_n^* follows the power law with exponent -1. Figure 3(b) shows the distributions of the population IETs with the observation period $T_o = 10^5$ and of the single-node IETs of a uniformly sampled node, obtained by the direct simulation of the model with the same parameters as in Fig. 3(a). We find that both IETs, $P_e(\Delta t)$ and $P_s(\Delta t)$, are close to the power law predicted theoretically.

The resource amount of a node x_i determines its activation probability a_i, i.e., $a_i \propto x_i$. A natural extension is to consider nonlinear preferential connections ($a_i \propto x_i^\alpha$) [25]. In some situations there is an advantage of large scale, in which accumulated resource enhances superlinearly the activity of a person ($\alpha > 1$). On the other hand, there may be saturation of the effect of the resources, causing sublinear activity ($\alpha < 1$). Figure 4 shows that the nonlinear preferential rule also

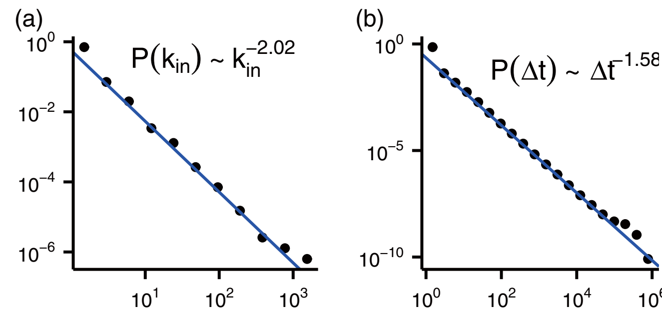

FIG. 4. Effect of nonlinear activation probability. (a) The degree distribution $P(k)$. (b) Single-node IET distribution $P_s(\Delta t)$ (log-log graphs). We use $\alpha = 2$, $\kappa = 0.0001$, $D = 1$, $N_A = 512$, and $N = 8192$.

produces the power-law distributions of node degree and IETs, but with different exponents. The power-law exponent is about 2 for the degree distribution and about 1.5 for IETs distribution if $\alpha = 2$. Our findings are consistent with empirical results of online communication in a web forum [25] and with correspondence patterns (waiting times) of famous scientists [7], reinforcing the intuition that some classes of communication processes are regulated by feedback mechanisms between information available and human activity.

In this Rapid Communication, we proposed a model of contact networks where the human dynamics are adaptively regulated by past interaction and exchange of resources. We analyzed the master equation of the model and found structural and temporal heterogeneities which are remarkable properties observed in many real-world systems. This revealed that these two types of heterogeneity can be observed in rich-clublike systems where the active individuals typically communicate with other active individuals, but occasionally connect to anyone in the network.

Despite the relative simplicity, the model provides a potential mechanistic understanding of the online communication dynamics, pointing to a number of unsolved questions, including the exact nature of the observed transition. Moreover, there are other important structural and temporal properties that our model cannot explain, such as community structure [30–32]. We therefore hope that it will stimulate future work, leading to deeper insights in the behavioral patterns of humans.

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